Common Derivatives

Fangyuan Lin
https://fangyuanlin2002.github.io/

Contents

0.1 Remarks

- Find it annoying to always forget the common multivariate derivatives? Me too.
- All variables are assumed to be vectors of appropriate dimensions and $\vec{x}$ is written as x for convenience.

0.2 Common Derivatives

0.2.1 Linear Form

Linear Form

Let

$$f(x) = a^T x.$$

Then

$$\nabla a^T x = a$$

Proof.

$$f(x) = \sum_i a_i x_i$$

$$\frac{\partial f}{\partial x_i} = a_i$$

□

0.2.2 Affine Function

Affine Function

Let

$$f(x) = Ax + b$$

Then

$$J_f(x) = A$$

Proof.

$$f_i(x) = \sum_j A_{ij}x_j + b_i$$
$$\frac{\partial f_i}{\partial x_j} = A_{ij}$$

□

0.2.3 Bilinear Form / Inner Product

Bilinear Form

Let

$$f(x, y) = x^T Ay$$

Then

$$\nabla_x f = Ay, \quad \nabla_y f = A^T x$$

Proof. Fix y. Then

$$f(x) = \sum_i x_i(Ay)_i$$
$$\frac{\partial f}{\partial x_i} = (Ay)_i$$

Fix x. Then

$$f(y) = \sum_i (x^T A)_i y_i$$
$$\frac{\partial f}{\partial y_i} = (x^T A)_i$$
$$\nabla_y f = (x^T A)^T = A^T x$$

□

0.2.4 Quadratic Form

Quadratic Form

Let

$$f(x) = x^T Ax$$

Then

$$\nabla f = (A + A^T)x$$

$$\begin{aligned} f(x) &= \sum_i x_i (Ax)_i \\ &= \sum_i x_i (\sum_j A_{ij} x_j) \\ &= \sum_i \sum_j x_i A_{ij} x_j \\ \frac{\partial f}{\partial x_k} &= \sum_{j \neq k} A_{kj} x_j + 2A_{kk} x_k + \sum_{i \neq k} A_{ik} x_i \\ &= (Ax)_k + (A^T x)_k \end{aligned}$$

0.2.5 Squared Norm

Squared Norm

Let

$$f(x) = \|x\|^2 = x^T x.$$

Then

$$\nabla f(x) = 2x$$

Proof. Note that his is a special case of quadratic form with $A = I$. Also one can consider:

$$\begin{aligned} f(x) &= \sum_i x_i^2 \\ \frac{\partial f}{\partial x_i} &= 2x_i \end{aligned}$$

□