Graduate Probability Theory and Stochastic Processes Notes

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Contents

Introduction

The journal of studying probability theory begins with some cliche. Sometimes, it's okay to do something boring.

1.1 Elementary Probability

A probability measure is a function $\mathbb{P}$ from subsets of a set Ω (sample space) to $[0, 1]$ with properties

1. $\mathbb{P}(\Omega) = 1$
2. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ whenever A and B are disjoint, i.e. $A \cap B = \emptyset$.

An example

Modelling roll of a fair die $\Omega = \{1, 2, \dots, 6\}$.
Let $\mathbb{P}(\{1\}) = \cdots = \mathbb{P}(\{6\}) = \frac{1}{6}$. We can get the probability of other subsets by addition.

Here is an example that is not-so-elementary.

Tossing coin

Toss a fair coin, we are interested in the number of tosses until we get the first head.
A natural way of modelling this is by letting $\Omega = \{1, 2, 3, \dots\}$

- $\mathbb{P}(\{1\}) = \frac{1}{2}$
- $\mathbb{P}(\{2\}) = \frac{1}{4}$
- $\mathbb{P}(n) = \frac{1}{2}^n$

$$\mathbb{P}(\text{takes odd \# tosses to see frist head}) = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \dots$$

What justifies this calculation? We need **countable** additivity, i.e. If $A_1, A_2, \dots$ are such that $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

1.2 Problem with Uncountable Additivity

So far, we have been able to calculate arbitrary subset of Ω. However, it was a big shock to mathematicians when they discovered that there was problem with that when Ω is uncountable.

Vitali Set

We can't always build natural probability measures that assign a probability to *every* subset of an uncountable.

- $\Omega =$ circle of radius 1. (Circumference 2π)
- Say ω', ω'' belong to the same family if it is possible to go from one to the other taking steps of unit length (unit length is defined by one radian away). This is an equivalence relation.
- The circle is partitioned into disjoint families.
- Each family is a countable dense subset of Ω.

Now let's consider a probability measure on the circle.

1. Suppose we can define a probability measure $\mathbb{P}$ on all subsets of Ω such that the probability of an interval is proportional to its length. (The idea of uniform probability measure on the circle.)
2. For each family, pick a point in the family to be the head of the family.
3. Set $A = \{\omega \in \Omega : \omega \text{ is the head of the family.}\}$
4. What is $\mathbb{P}(A)$? Well, it turns out that the supposition that we can define such a probability measure must be false.
5. To see that, let
$$B_i = \{\omega \in \Omega : \omega \text{ is } i \text{ units counter-clockwise from the head of its family}\}$$
and let
$$C_i = \{\omega \in \Omega : \omega \text{ is } i \text{ units clockwise from the head of its family}\}$$
remark. We need to make uncountable choices to pick the heads. This is a set theory problem. We actually need to appeal to the axiom of choice.
6. Next, note that $\mathbb{P}$ is rotation invariant so $\mathbb{P}(A) = \mathbb{P}(B_i =) = \mathbb{P}(C_i)$ for all i. Note that $A, B_1, B_2, \dots, C_1, C_2, \dots$ pairwise disjoint, because 2π is irrational. For B_i and C_j to overlap, then somehow we need $i + j = 0 \mod 2\pi$ which is impossible.
7. Then
$$1 = \mathbb{P}(\Omega) = \mathbb{P}(A \cup B_1 \cup B_2 \cup \dots \cup C_1 \cup C_2 \cup \dots),$$
which is impossible.

remark. There's actually a deep theorem saying that there exists no probability measure on the circle that assigns zero probability to each point.

To remedy this, we realize that we can only assign a probability to some subsets of the sample space.

Sometimes, people describe this phenomenon as: *we don't always have complete information about the world.* By this, they mean there may be some events to which we cannot assign probabilities.

Sigma-Fields and Measures

2.1 Sigma-Fields

With the example of Vitali circle in mind, we have to restrict the definition of probability measures to a subset of the power set of the sample space. This brings us to the definition of sigma-field, a space with properties needed for our theory to work.

σ-field

Define a σ-algebera or σ-field in Ω as a nonempty collection $\mathcal{F}$ of subsets of Ω such that if $E_1, E_2, \dots \in \mathcal{F}$, then so are E^C and $\bigcup_k E_k$ (and so is $\bigcap_k E_k$ by de Morgan)

- Additionally, $\Omega \in \mathcal{F}$
- A sigma-field is also called a field, or sigma-algebra.

Examples of Sigma-Fields

The following spaces are sigma-fields and it's easy to very this claim.

- The power set:
$$\mathcal{F} = 2^\Omega$$
- The trivial sigma-field:
$$\mathcal{F} = \{\emptyset, \Omega\}$$
-
$$\Omega = \{1, 2, 3\} \quad \mathcal{F} = \{\emptyset, \{1\}, \{2, 3\}, \Omega\}$$
-
$$\mathcal{F} = \{B \subset \Omega : B \text{ is countable or } B^C \text{ is countable.}\}$$

Following lemma shows a nice property of sigma-field.

Lemma: Intersection of sigma-fields is a sigma-field

If I is any index set and $\{\mathcal{F}_i : i \in I\}$ are σ-fields over Ω, then $\bigcap_{i \in I} \mathcal{F}_i$ is a σ-field.

Proof. 1. Ω: Each $\mathcal{F}_i$ has Ω.

2. Closed under complement: If $E \in \bigcup_i \mathcal{F}_i$, then $E \in \mathcal{F}_i$ for all i. Then $E^C \in \mathcal{F}_i$ for all i.

3. Closed under countable union: Similar argument.

□

Now, we naturally have the following proposition. Isn't it reminiscent of topology? (In case you are not sure, it is.)

Proposition:Uniquely generated minimal sigma-field

Let $\mathcal{F} \subset 2^\Omega$ be any collection of subsets of Ω. There is a unique smallest σ-field $\sigma(\mathcal{F})$ that contains $\mathcal{F}$.

- It is called the σ-field generated by $\mathcal{F}$.

Proof.

$$\sigma(\mathcal{F}) = \bigcap \{\mathcal{A} : \mathcal{A} \text{ is a } \sigma\text{-field over } \Omega \text{ such that } F \subset \mathcal{A}\}.$$

□

Here's another proposition that will be useful.

Lemma: condition for generators to be the same

Let $\mathcal{F}_1, \mathcal{F}_2 \subset 2^\Omega$. Then

$$\sigma(\mathcal{F}_1) = \sigma(\mathcal{F}_2)$$

if and only if

$$\mathcal{F}_1 \subset \sigma(\mathcal{F}_2) \quad \& \quad \mathcal{F}_2 \subset \sigma(F_1)$$

Proof. Well

- Suppose $\sigma(\mathcal{F}_1) = \sigma(\mathcal{F}_2)$, then

$$\mathcal{F}_i \subset \sigma(\mathcal{F}_i) = \sigma(F_j)$$

for $(i, j) = \{(1, 2), (2, 1)\}$

- Suppose $\mathcal{F}_1 \subset \sigma(\mathcal{F}_2) \quad \& \quad \mathcal{F}_2 \subset \sigma(F_1)$. Since $\mathcal{F}_1 \subset \sigma(\mathcal{F}_2)$, $\sigma(\mathcal{F}_2)$ is then a sigma-field containing $\mathcal{F}_2$. Then $\sigma(\mathcal{F}_1) \subset \sigma(\mathcal{F}_2)$ by the minimality of $\sigma(\mathcal{F}_1)$. We have the other direction by swapping 1 and 2.

□

We now involve some topology in our discussion of measure theory.

Borel Sigma Field

Let X be a topological space (e.g. Euclidean space $\mathbb{R}^d$). The Borel σ-field $\mathcal{B}(X)$ is the σ-field generated by the open subsets in X.

$$\mathcal{B}(X) = \sigma(\{\text{open subsets of } X\}).$$

Events in $\mathcal{B}(X)$ are called Borel sets.

- Here is a useful fact. For $X = \mathbb{R}^d$, every open set U is countable union of open balls
$$U = \bigcup_{i=1}^{\infty} B(x_i, r_i)$$
Therefore,
$$\mathcal{B}(\mathbb{R}^d) = \sigma\{\text{open balls in } \mathbb{R}^d\}.$$

2.2 Measures

Additivity is an essential assumption on probability measures. This is analogous to the linearity of linear functions.

Countable Additivity

A function $\mu : \mathcal{F} \to [0, \infty]$ is called a measure if it is countably additive. i.e., if If $E_1, E_2, E_3, \dots \in \mathcal{F}$ are all disjoint, then

$$\mu(\bigsqcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n).$$

The triplet $(\Omega, \mathcal{F}, \mu)$ is then called a measure space. If $\mu(\Omega) < \infty$, we say μ is a finite measure. If $\mu(\Omega) = 1$, we say μ is a probability measure and $(\Omega, \mathcal{F}, \mu)$ is a probability space. The number 1 is a special number, needless to say lol.

Example of measures

- The constant measures $\mu \equiv 0$ or $\mu \equiv \infty$ (on any $(\Omega, \mathcal{F})$).
- Point mass: on $(\Omega, 2^\Omega)$, fix a point $\omega_0 \in \Omega$, and define by
$$\delta_{\omega_0} : 2^\Omega \to \{0, 1\}$$
by
$$\delta_{\omega_0}(E) = \begin{cases} 1 & \textbf{if } \omega_0 \in E. \\ 0 & \text{if } \omega_0 \not\in E. \end{cases}$$
- To verify its additivity, note that if $\{E_n\}_{n=1}^{\infty}$ are disjoint, then ω_0 is in at most 1 of the E_n. The next step is to consider the two cases: the case where there is some E_n that contains ω_0 and the case where there is no E_n that contains ω_0.

Positive Scalar Multiple of a Measure is still a Measure. Countable Sum of Measures is still a Measure.

If μ is a measure, so is $\alpha\mu$ for any $\alpha \geq 0$.

$$(\alpha\mu)(E) = \alpha \cdot \mu(E).$$

If $\{\mu_j\}_{j=1}^{\infty}$ is a countable set of measures and $\mu_j \geq 0$ and is allowed to take infinite value (for convergence considerations), then

$$\mu = \sum_{j=1}^{\infty} \mu_j$$

is a measure.

Proof. If $\{E_i\}_{i=1}^{\infty}$ are disjoint events in $\mathcal{F}$, then

$$\begin{aligned}
\mu(\bigsqcup_{i=1}^{\infty} E_i) &= \sum_{j=1}^{\infty} \mu_j(\bigsqcup_{i=1}^{\infty} E_i) \\
&= \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} \mu_j(E_i) \right) \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu_j(E_i). \\
&= \sum_{i=1}^{\infty} \mu(E_i)
\end{aligned}$$

The last equality is by Tonelli's theorem since we have positivity of the μ_j's. □

Construction of Discrete Probabilities

$$\mu = \sum_{j=1}^{\infty} p_j \delta_{\omega_j}$$

for some $\omega_j \in \Omega, p_j \geq 0$. This is a weighted sum of indicators.
If $\sum_{j=1}^{\infty} p_j = 1$, then this is a probability measure

$$\mu(\Omega) = \sum_j p_j = 1.$$

If $\{\omega_j\}_{j=1}^{\infty}$ is all of Ω, this is called discrete probability:

$$\mu(E \in \mathcal{F}) = \sum_{j:\omega_j \in E} p_j = \sum_{\omega \in E} \mu(\{\omega\})$$

The Very Basic Properties of Measures

1. Monotonicity: If $A, B \in \mathcal{F}$ and $A \subseteq B$, $\mu(A) \leq \mu(B)$.

   Proof.
   $$B = (B - A) \sqcup A$$
   □

2. Some algebra: If $A, B \in \mathcal{F}$,
   $$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$$
   $$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

3. Subadditivity: If $\{B_n\}_{n=1}^{\infty}$ are in $\mathcal{F}$, then $\mu(\bigcup_{n=1}^{\infty}) \leq \sum_{n=1}^{\infty} \mu(B_n)$

   Proof. Consider $A_1 = B_1, A_2 = B_2 - B_1, ..., A_n = B_n - (B_1 \cup \cdots \cup B_{n-1})$. They form the same union. □

2.3 Finitely Additive Measures

Measures are often hard to construct, so we can start with a weaker notion of measure that we can build up to get a measure.

Premeasurable Space

A pair $(\Omega, \mathcal{A})$ is a premeasruable space if $\mathcal{A}$ is a *field* over Ω. A countably additive function
$$\mu : \mathcal{A} \to [0, \infty]$$
is called a premeasure.
If we assume that χ is only finitely-additive, i.e.
$$\chi(A \sqcup B) = \chi(A) + \chi(B),$$
We call χ a *finitely-additive measure.*

- The difference between a field and a σ-field is that a field is only assumed to be closed under *finite* union.
- The difference between a premeasure and a measure it's just the structure of their domains.

Our first objective is to construct a finitely additive measure on a field.

Superadditivity of Finitely additive Measure

Let $(\Omega, \mathcal{A}, \chi)$ be a finitely-additive measure space. If $\{A_i\}_{i=1}^{\infty}$ are disjoint in $\mathcal{A}$ and it so happens that

$$A = \bigsqcup_{i=1}^{\infty} A_i \in A,$$

then

$$\chi(A) \geq \sum_{i=1}^{\infty} \chi(A_i)$$

Proof. Here is a very important technique: when dealing something infinite, start with something finite. Let $n \in \mathbb{N}$ be arbitrary,

$$\bigsqcup_{i=1}^{n} A_i \in \mathcal{A}$$

because $\mathcal{A}$ is a field. It is also a subset of A so but monotonicity,

$$\chi(\bigsqcup_{i=1}^{n} A_i) \leq \chi(A)$$

Now, we complete the proof by taking the limit as $n \to \infty$. □

We will see an example of how this inequality can be strict.

2.3.1 A Borel Field

One possible natural generating set for the Borel σ-field $\mathcal{B}(\mathbb{R})$ is

$$b_{(]} = \{(a, b] : -\infty \leq a \leq b \leq \infty\}$$

What about the field gerated by these intervals? It is

$$\mathcal{A} \supset \{\text{finite disjoint unions of intervals in } b_{(]}\}$$

It's clear (especially if we visualize this) that the above subset satisfies the field axioms (closed under finite union and complement),
so $\mathcal{A}$ is actually equal to finite disjoint unions of intervals in $b_{(]}$.

2.3.2 Semi-Algebras of Sets

We now introduce an even weaker structure on Ω.

Semi-Algebra

A collection $\mathcal{S} \subset 2^{\Omega}$ if called a semi-algebra or elementary class if

- $$\emptyset \in \mathcal{S}$$
- If $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$.

- If $\mathcal{A} \in \mathcal{S}$, then A^C is a finite disjoint union of elements from $\mathcal{S}$.

The canonical example of semi-algebra is

$$\mathcal{S} = \{(a, b] : -\infty \leq a \leq b \leq \infty\}.$$

Field Generated by Semi-Algebra

If $\mathcal{S}$ is a semi-algebra over Ω, then the field generated by $\mathcal{S}$, $\mathcal{A}(\mathcal{S})$, is equal to

$$DU(\mathcal{S}) := \{\text{All finite disjoint unions of sets from } \mathcal{S}\}$$

Proof.

- First of all,
$$\mathcal{S} \subset DU(\mathcal{S}) \subset \mathcal{A}(\mathcal{S})$$
because a field is automatically closed under finite union. It suffices to show that $DU(\mathcal{S})$ is a field itself.

- To check closure under finite intersection,
$$D = \bigsqcup_{i=1}^{n} A_i, E = \bigsqcup_{j=1}^{m} B_i \in DU(\mathcal{S})$$
$$D \cap E = (\bigsqcup_{i=1}^{n} A_i) \cap (\bigsqcup_{j=1}^{m} B_i) = \bigsqcup_{i,j} (A_i \cap B_j) \in DU(\mathcal{S})$$
Each of $A_i \cap B_j \in S$ since definition of semi-algebra, and different pairs are disjoint.

- Closure under complement:
$$D^C = \bigcap_i A_i^C$$
Each of the A_i^C is a finite disjoint union $\bigsqcup_{j_i} C_{j_i}$of elements in $\mathcal{S}$.
$$\bigcap_i A_i^C = \bigcap_i (\bigsqcup_{j_i} C_{j_i}) = \bigsqcup_{j_i, 1 \leq i \leq m} (\bigcap_{i,j_i} C_{j_i}) \in DU(\mathcal{S})$$

□

Let $\mathcal{S}$ be a semi-algebra over Ω. Let

$$\chi : \mathcal{S} \to [0, \infty]$$

be finitely additive. Then there is a *unique* extension of χ to a finitely-additive measure on $A(\mathcal{S})$, defined by

$$A = \bigsqcup_{i=1}^{n} E_i \implies \chi(A) := \sum_{i=1}^{n} \chi(E_i)$$

Proof. This identity must hold if χ is a finitely additive measure. Therefore, it uniquely defines the extension. We have to show that this is well-defined, i.e. $\chi(A)$ cannot possibly be multi-valued.

- Suppose there are two ways to partition the set A,

$$A = \bigsqcup_{i=1}^{n} E_i = \bigsqcup_{j=1}^{m} F_j$$

- Note that

$$E_i = \bigsqcup_j E_j \cap F_j$$

$$\chi(E_i) = \sum_j \chi(E_i \cap F_j), \quad \chi(A) = \sum_{i=1}^{n} \sum_{j=1}^{m} \chi(E_i \cap F_j)$$

□

2.3.3 Stieljes Premeasures on the Borel Field

Let

$$F : \mathbb{R} \to \mathbb{R}$$

be any non-decreasing function. On the semi-algebra

$$b_{(]} = \{(a,b] : -\infty \le a \le b \le \infty\},$$

define

$$\chi_F((a,b]) = F(b) - F(a) \ge 0.$$

We claim that χ_F is additive on the semi-algebra $d_{(]}$:

Proof.

$$\begin{aligned}
(a,b] &= (a,c] \sqcup (c,d], \quad a < c < b \\
\chi_F(a,b] &= F(b) - F(a) \\
&= (F(c) - F(a)) + (F(b) - F(a)) \\
&= \chi_F((a,c]) + \chi_F((c,b])
\end{aligned}$$

□

By the previous proposition, we know that χ_F extends to a finitely-additive measure on

$$\mathcal{A}(b_{(]}) = \mathcal{B}_{(]}(\mathbb{R}).$$

Now let's answer the question: Is it a premeasure? That is, is it countably additive?

Necessity of Right Continuity

Fix $a \in \mathbb{R}$,

$$(a, a+1] - \bigsqcup_{n=1}^{\infty} (a + \frac{1}{n+1}, a + \frac{1}{n}] \in \mathcal{B}_{(]}(\mathbb{R})$$

Now

$$\begin{aligned} \chi_F((a, a+1]) &= F(a+1) - F(a) \\ \sum_{n=1}^{\infty} \chi_F((a + \frac{1}{n+1}, a + \frac{1}{n}]) &= \sum_{n=1}^{\infty} F(a + \frac{1}{n+1}) - F(a + \frac{1}{n}) \\ &= \lim_{m \to \infty} F(a+1) - F(a + \frac{1}{m}) \\ &= F(a+1) - F(a+) \end{aligned}$$

where

$$F(a+) := \lim_{\epsilon \downarrow a} F(a + \epsilon)$$

Therefore, χ_F is not countably additive if $F(a+) \neq F(a)$. This shows that for a finitely-additive measure, it's possible to have $\chi(A) > \sum_{i=1}^{\infty} \chi(A_i)$ as we remarked previously.

χ_F is a premeasure iff F is right-continuous

Theorem 2.1. *The finitely-additive measure χ_F is a premeasure (i.e. countably additive) on $\mathcal{B}(\mathbb{R})$ iff F is right-continuous on $\mathbb{R}$, i.e.*

$$\lim_{\delta \to 0} F(a + \delta) = F(a)$$

Finitely additive on a field and countably subadditive on semi-algebra means countably additive on field.

Let $\mathcal{S} \subseteq 2^{\Omega}$ be a semi-algebra. A finitely-additive measure

$$\chi : \mathcal{A}(\mathcal{S}) \to [0, \infty]$$

is a premeasure iff it is countably subadditive on $\mathcal{S}$:

$$E = \bigsqcup_{j=1}^{\infty} E_j \in S \implies \chi(E) \leq \sum_{j=1}^{\infty} \chi(E_j)$$

Proof.

- Premeasures are countably additive.
- Finitely-additive measures are always superadditive, so it suffices to prove that χ is countably subadditive on $\mathcal{A} = \mathcal{A}(\mathcal{S})$.
- Let

$$A = \bigsqcup_{n=1}^{\infty} A_n$$

where A and A_n are sets in the field $\mathcal{A}$. Now we decompose them into sets from $\mathcal{S}$. Let

$$A = \bigsqcup_{j=1}^{n} E_j, \quad A_n = \bigsqcup_{i=1}^{N_n} E_i^n$$

Now let's apply some set algebra,

$$E_j = \bigsqcup_{n} (A_n \cap E_j) = \bigsqcup_{n} \bigsqcup_{i=1}^{N_n} E_i^n \cap E_j$$

By our assumption of χ being subadditive on the semi-algbera $\mathcal{S}$,

$$\chi(E_j) \leq \sum_n \sum_i \chi(E_i^n \cap E_j)$$

Now

$$\begin{aligned} \chi(A) &= \sum_{j=1}^{N} X(E_j) \\ &\leq \sum_{j=1}^{N} \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \chi(E_i^n \cap E_j) \\ &\leq \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \sum_{j=1}^{N} \chi(E_i^n \cap E_j) \\ &= \sum_{n=1}^{\infty} \chi(A_n) \end{aligned}$$

□

We now that the Stieljes construction with a right-continuous F

$$\chi_F : \mathcal{A}(d_{(]}) = \mathcal{B}_{(]}(\mathbb{R}) \to [0, \infty)$$

is indeed a premeasure by showing that it is countably subadditive on the semi-algbera $b_{(]} = \{(a, b]\}$. Let's assume that $a, b < \infty$ for convenience. Let

$$(a, b] = \bigsqcup_{j=1}^{\infty} (a_j, b_j].$$

Compactness is a useful concept for us to reduce a countable situation to a finite situation. Let's let consider a closed interval

$$[a + \delta, b] \subset (a, b] = \bigsqcup_{[} j = 1]^{\infty} (a_j, b_j].$$

Compactness is about open covers, so let's modify the upper bound of $(a_j, b_j]$ a little bit:

$$[a + \delta, b] \subset (a, b] = \bigsqcup_{j=1}^{\infty} (a_j, b_j] \subset \bigsqcup_{j=1}^{\infty} (a_j, b_j + \delta_j)$$

By compactness,

$$[a+\delta, b] \subset \bigsqcup_{j=1}^{N} (a_j, b_j+\delta_j) \quad \text{for some } N < \infty.$$

χ_F is a finitely additive measure over the field of unions.

$$\begin{aligned}
\chi_F(a+\delta, b] &\leq \sum_{j=1}^{N} \chi_F(a_j, b_j+\delta_j] \leq \sum_{j=1}^{\infty} \chi_F(a_j, b_j+\delta_j] \\
&= \sum_{j=1}^{\infty} \chi_F(a_j, b_j] + \chi_F(b_j, \delta_j] \\
\chi_F(a+\delta, b] &\leq \sum_{j=1}^{\infty} \chi_F(a_j, b_j] + \sum_{j=1}^{\infty} \chi_F(b_j, \delta_j]
\end{aligned}$$

$\sum_{j=1}^{\infty} \chi_F(b_j, \delta_j]$ can be arranged to be arbitrarily small. For example, for any $\epsilon > 0$, choose $\delta_j > 0$ such that $F(b_j+]delta_j) - F(b_j) < \frac{\epsilon}{2_j}$. We can do this by the right continuity of F.
On the other hand,

$$\chi_F(a+\delta, b] = F(b) - F(a+\delta) \to F(b) - F(a)$$

as $\delta \to 0$ and the right hand side does not change when we take the limit. Therefore, we have shown that

$$\chi_F \text{ is a premeasure on the Borel } \textit{field } \mathcal{B}_{(]}(\mathbb{R}).$$

If we choose

$$F(x) = x, \quad \chi(a, b] = b - a,$$

then we call it *Lebesgue premeasure*. Notably, the Lebesgue premeasure is translation-invariant.
Our next goal is to extend our premeasure on the field to a full measure on a sigma-field.